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1980 J. Phys. A: Math. Gen. 13 1737

(http://iopscience.iop.org/0305-4470/13/5/031)

J. Phys. A: Math. Gen. 13 (1980) 1737–1740. Printed in Great Britain

Recursive calculation of axially symmetric stationary Einstein fields

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Received 26 September 1979, in final form 4 January 1980

Abstract. A simple recursion formula is presented for calculating stationary axisymmetric (asymptotically flat) Einstein fields with any number of constants. It generates Kerr (Schwarzschild) particles from the vacuum (Minkowski space), and stationary asymptotically flat solutions from static ones.

In a previous Letter (Neugebauer 1979) it was shown that Bäcklund transformations can be used to generate stationary axisymmetric gravitational fields with any number of constants. The point is that the Bäcklund transformation method essentially involves algebraic manipulations. In this paper the algebraic procedure is reduced to a simple recursion formula, which, working as a nonlinear creation operator, generates, from a given solution, new gravitational fields with additional parameters. For instance, applied to the flat Minkowski space (the 'vacuum') it creates a Kerr 'particle' (Neugebauer and Kramer 1980). Within the purely static Weyl class successive applications lead from flat space to a superposition of any number of Schwarzschild solutions. Furthermore, the recursion formula can be exploited to construct, for each static asymptotically flat spacetime, a corresponding stationary asymptotically flat spacetime.

Consider the system of total differential equations

$$\begin{aligned} d\psi &= (f+g)^{-1}(\psi\, dg + \chi\gamma^{1/2} g_{,1}\, dx^1 + \chi\gamma^{-1/2} g_{,2}\, dx^2), \\ d\chi &= (f+g)^{-1}(\chi\, df + \psi\gamma^{1/2} f_{,1}\, dx^1 + \psi\gamma^{-1/2} f_{,2}\, dx^2), \\ d\gamma &= W^{-1}(\gamma-1)(\gamma W_{,1}\, dx^1 + W_{,2}\, dx^2) \end{aligned} \tag{1}$$

for the potentials $\psi(x^1, x^2)$, $\chi(x^1, x^2)$, $\gamma(x^1, x^2)$. It is completely integrable, if f, g, W are solutions of the complexified Ernst equations

$$\begin{aligned} (Wf_{,1})_{,2} + (Wf_{,2})_{,1} &= 4W(f+g)^{-1} f_{,1} f_{,2}, \\ (Wg_{,1})_{,2} + (Wg_{,2})_{,1} &= 4W(f+g)^{-1} g_{,1} g_{,2}, \\ W_{,1,2} &= 0. \end{aligned} \tag{2}$$

Gravitational fields are special solutions of (2) with $f = \bar{g}$, $W = \bar{W}$ and $\overline{x^2} = x^1 = \rho + iz$, where ρ and z are cylindrical coordinates. A bar denotes complex conjugation.

Let f^0, g^0, W^0 be a known solution of (2). Using (1) and following the diagram

$$(f^0, g^0, W^0) \to (\psi_k^0, \chi_k^0, \gamma_k^0) \to (\psi_m^l, \chi_m^l, \gamma_m^l) \to (f^l, g^l, W^l) \tag{3}$$

calculate a countable set of potentials ψ_k^0, χ_k^0, γ_k^0 $(k=1,2,3\ldots)$ with different integration constants indicated by the index k. Then the recursion formulae

$$\psi_k^{i+1}=\lambda_k^i\frac{(\gamma_{k+1}^i)^{1/2}\chi_{k+1}^i+\psi_{k+1}^i}{(\gamma_1^i)^{1/2}\chi_1^i+\psi_1^i},$$

$$\chi_k^{i+1}=-\lambda_k^i\frac{(\gamma_{k+1}^i)^{1/2}\psi_{k+1}^i+\chi_{k+1}^i}{(\gamma_1^i)^{1/2}\psi_1^i+\chi_1^i},$$

$$\gamma_k^{i+1}=\frac{\gamma_{i+k+1}^0}{\gamma_{i+1}^0},\qquad \gamma_0^0=1,\qquad \lambda_k^i=\left(\frac{1-\gamma_{i+1}^0}{1-\gamma_{i+k+1}^0}\right)^{1/2} \tag{4}$$

implicitly define new potentials ψ_m^l, χ_m^l, γ_m^l from which new solutions f^l, g^l, W^l of (2) can be derived (l indicates the number of iterations). It is not hard to perform this last step of (3). Via recursion formula (4) the potentials ψ_m^l, χ_m^l are functionals of γ_{l+m}^0: $\psi_m^l\{\gamma_{l+m}^0\}$, $\chi_m^l\{\gamma_{l+m}^0\}$. Let us choose the integration constant in γ_{l+m}^0 in such a way that $\gamma_{l+m}^0=\gamma_l^0$ and therefore $\gamma_m^l=\gamma_{l+m}^0/\gamma_l^0=1$. For this special value of γ_m^l we obtain special values $\psi_m^l\{\gamma_m^l=1\}$, $\chi_m^l\{\gamma_m^l=1\}$ of the potentials ψ_m^l, χ_m^l. Consider the first and second equations of (1), which link the potentials ψ_m^l, χ_m^l with the corresponding Ernst functions. Setting $\gamma_m^l=1$ in these equations and integrating them we find the wanted connection

$$\psi_m^l\{\gamma_m^l=1\}=\alpha_m^l g^l+\beta_m^l,\qquad \chi_m^l\{\gamma_m^l=1\}=\alpha_m^l f^l-\beta_m^l, \tag{5}$$

where α_m^l and β_m^l are integration constants. (Note that the Ernst functions of a fixed recursion order l do not depend on a lower index m. This follows from the recursion formula and the fact that initial potentials ψ_k^0, χ_k^0 with different values of the index k are calculated from the same Ernst potentials g^0, f^0.) The decoupled field W^l can explicitly be calculated from W^0. (Note that generally $W^0=A(\rho+\mathrm{i}z)+\overline{A(\rho+\mathrm{i}z)}$, $\gamma_k^0=(1+\mathrm{i}L_k\bar{A})(1-\mathrm{i}L_kA)^{-1}$, with L_k real constants, hold.) Algebraic recursion formulae for the other metric coefficients are also available.

The algorithm (3) immediately results from the Bäcklund transformation theory outlined in the previous paper. Indeed, combining the ψ–χ equations in (1) one obtains the total Riccati equation for $\alpha=-\gamma^{1/2}\psi\chi^{-1}$ used there. The recursion formula (4) is a consequence of the algebraic α–γ transformation formalism which was made in a system with the help of a graph technique.

As a first example we apply the procedure (3) to the flat space solution $f^0=g^0=1$, $W^0=|x^1+\overline{x^1}||m\cos\phi+\mathrm{i}x^1|^{-2}$ ($x^1=\rho+\mathrm{i}z$; m, ϕ integration constants) and choose as solutions of (1) the potentials

$$\psi_1^0=\mathrm{e}^{2\mathrm{i}\phi},\qquad \psi_2^0=\psi_3^0=\chi_1^0=1,\qquad \chi_2^0=-\chi_3^0=-\mathrm{e}^{-\mathrm{i}\phi},$$

$$\gamma_1^0=\gamma_2^0\frac{m\cos\phi+\mathrm{i}\overline{x^1}}{m\cos\phi-\mathrm{i}x^1},\qquad \gamma_2^0=\frac{m\cos\phi+\mathrm{i}x^1}{m\cos\phi-\mathrm{i}\overline{x^1}},\qquad \gamma_3^0=\gamma_2^0\frac{1+\mathrm{i}K\overline{x^1}}{1-\mathrm{i}Kx^1}. \tag{6}$$

After a double step in (4) we put $K=0$ $(\gamma_1^2=1)$. Then according to (5) a possible choice is

$$f=f^2=\chi_1^2\{\gamma_1^2=1\}=\overline{\psi_1^2\{\gamma_1^2=1\}}\qquad (\alpha_1^2=1,\beta_1^2=0),$$

and we have

$$f=\frac{\mathrm{e}^{\mathrm{i}\phi}r_1+\mathrm{e}^{-\mathrm{i}\phi}r_0-2m\cos\phi}{\mathrm{e}^{\mathrm{i}\phi}r_1+\mathrm{e}^{-\mathrm{i}\phi}r_0+2m\cos\phi}, \tag{7}$$

where $r_1 = |m \cos \phi + z - \mathrm{i}\rho|$ and $r_0 = |m \cos \phi - z + \mathrm{i}\rho|$. This is the Kerr solution with the mass m and the rotation parameter $l = m \cos \phi$ in Weyl coordinates $\{\rho, z\}$ (cf Neugebauer and Kramer 1980). The infinite-parameter group K used by Kinnersley *et al* is a subgroup of the Bäcklund transformation group P introduced in our previous paper. The application of symmetry operations of K yields the extreme Kerr solution (Hoenselaers *et al* 1979). It should be remarked that Herlt (1978) was able to generate the Kerr solution from a complex van Stockum solution.

The choice $\psi_k^0 = 1$, $\chi_k^0 = (-1)^{k+1}$ ensures that the solutions generated from the flat space solution $f^0 = g^0 = 1$ are static. After $2N$ iteration steps we obtain a (real) Ernst potential $f^{2N} = \psi_1^{2N} = \chi_1^{2N}$, which describes a superposition of N Schwarzschild masses m_k $(k = 1, 2 \ldots N)$ distributed along the z axis. This result seems to coincide with the static multi-soliton solution of Belinsky and Zakharov (1979). (The two-soliton *ansatz* and the first recursion double step lead to the Kerr solution.) The superposition of two Kerr solutions will be published elsewhere (Kramer and Neugebauer 1980). If f^0, g^0, W^0 is a (static) Weyl solution ($f^0 = g^0 = \overline{f^0}$), the integrals of (1) are easy to find. This holds for the Papapetrou class and van Stockum class ($g^0 = 0$) too (Neugebauer 1979). Starting with an arbitrary static solution and calculating ψ_1^2, χ_1^2 we are led to the stationary solution

$$f = -\frac{\bar{A}Br_1 + ABr_0 - 2A\bar{B}m \cos \phi}{\bar{A}Cr_1 + ACr_0 + 2A\bar{C}m \cos \phi}, \tag{8}$$

where

$$\begin{aligned} A &= \cos(\phi - \mathrm{i}\Phi_1) - \mathrm{i} \sin(\phi + \mathrm{i}\Phi_1), \\ B &= \cos(\tfrac{1}{2}\phi - \mathrm{i}\Phi) - \mathrm{i} \sin(\tfrac{1}{2}\phi + \mathrm{i}\Phi), \\ C &= \mathrm{i} \sin(\tfrac{1}{2}\phi - \mathrm{i}\Phi) - \cos(\tfrac{1}{2}\phi + \mathrm{i}\Phi). \end{aligned} \tag{9}$$

The symbols r_0, r_1, m and ϕ were explained in the first example.

The real function $\Phi_1(x^1, x^2 = \overline{x^1})$ ($x^1 = \rho + \mathrm{i}z$) follows from the (real) axisymmetric harmonic function $\Phi(x^1, x^2 = \overline{x^1})$ by means of the Backlund transformation

$$\Phi_{1,1} = \Phi_{,1}\left(\frac{m \cos \phi + \mathrm{i}\overline{x^1}}{m \cos \phi - \mathrm{i}x^1}\right)^{1/2}, \qquad \Phi_{1,2} = \overline{\Phi_{1,1}}. \tag{10}$$

As special cases f contains the Kerr solution ($\Phi = \Phi_1 = 0$; cf (7)), any static solution ($m = 0$, $\phi = 0$), a special Papapetrou solution ($m = 0$), and the superposition of any static solution $\mathrm{e}^{2\Phi}$ with the Schwarzschild solution ($\phi = 0$). Consider the static case ($\phi = 0$). Because each of the gravitational potentials of isolated sources consists of a mass term and higher multipoles, we can build up any wanted static field from the mass m of the Schwarzschild solution and correcting higher multipoles of Φ. In this way we avoid NUT-like singularities and obtain for each static asymptotically flat solution a corresponding stationary asymptotically flat solution (8).

The recursion formula (4) solves the algebraic problem of the Backlund transformation theory of axisymmetric stationary gravitational fields. Since the analytic problem of determining the initial values ψ_k^0, χ_k^0, γ_k^0 is solvable for the large Weyl class (Neugebauer 1979), the successively calculated solutions f^l, g^l, W^l ($l = 2, 4, 6, \ldots$) involve an arbitrary harmonic function as well as an arbitrary number of integration constants. What remains to be done is to adapt the countable set of integration constants in ψ_k^0, χ_k^0, γ_k^0 ($k = 1, 2, 3, \ldots$) to special physical problems. The examples given confirm our belief—in which we agree with Hoenselaers *et al* (1979)—that harmonic function plus integration constants are sufficient to characterise the mass distribution and the angular momentum distribution of an arbitrary axisymmetric stationary asymptotically flat Einstein field.

I am greatly indebted to my colleagues Drs Kramer and Herlt for many interesting discussions.

Note added in proof. In the meantime I have found the solution of the recursion formulae (4) for an arbitrary number of recursion steps. This solution leads to a simple determinant expression for the Ernst function $f = \bar{g}$ (Neugebauer G 1980 *J. Phys. A: Math. Gen.* **13** L19–21. Note that there is a misprint: equation (7) should read $\gamma_0 = 1$, $\alpha_0 = -\bar{f}_0/f_0$.) At present the determinant expression seems to be the most comprehensive (explicitly given) solution describing a spinning mass distribution.

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